

# FINITE GRÖBNER BASIS ALGEBRAS WITH UNSOLVABLE NILPOTENCY PROBLEM AND ZERO DIVISORS PROBLEM

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**ABSTRACT.** This work presents a sample construction of an algebra with the ideal of relations defined by a finite Gröbner basis for which the question whether a given element is nilpotent is algorithmically unsolvable. This gives a negative answer to a question raised by Latyshev.

## 1. INTRODUCTION

The word equality problem in finitely presented semigroups (and in algebras) cannot be algorithmically solved. This was proved in 1947 by Markov ([Ma]) and independently by Post([Po]). In 1952 Novikov constructed the first example of the group with unsolvable problem of word equality (see [N1] and [N2]).

In 1962 Shirshov proved solvability of the equality problem for Lie algebras with one relation and raised a question about finitely defined Lie algebras (see [Sh]).

In 1972 Bokut settled this problem. In particular, he showed the existence of a finitely defined Lie algebra over an arbitrary field with algorithmically unsolvable identity problem ([Bo]).

A detailed overview of algorithmically unsolvable problems can be found in [BK].

Otherwise, some problems become decidable if a finite Gröbner basis defines a relations ideal. In this case it is easy to determine whether two elements of the algebra are equal or not (see [Be]).

Gröbner bases for various structures are investigated by the Bokut school in Guangzhou ([BC]).

In his work, Piontkovsky extended the concept of obstruction, introduced by Latyshev (see [Pi1], [Pi2], [Pi3], [Pi4]).

Latyshev raised the question concerning the existence of an algorithm that can find out if a given element is either a zero divisor or a nilpotent element when the ideal of relations in the algebra is defined by a finite Gröbner basis.

Similar questions for monomial automaton algebras can be solved. In this case the existence of an algorithm for nilpotent element or a zero divisor was proved by Kanel-Belov, Borisenko and Latyshev [KBBL]. Iyudu showed that the element property of being one-sided zero divisor is recognizable in the class of algebras with a one-sided limited processing (see [I1], [I2]). It also follows from a solvability of a linear recurrence relations system on a tree (see [KB]).

An example of an algebra with a finite Gröbner basis and algorithmically unsolvable problem of zero divisor is constructed in [IP].

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In the present paper we construct an algebra with a finite Gröbner basis and algorithmically unsolvable problem of nilpotency. We also provide a shorter construction for the zero divisors question.

For these constructions we simulate a universal Turing machine, each step of which corresponds to a multiplication from the left by a chosen letter.

Thus, to determine whether an element is a zero divisor or is a nilpotent, it is not enough for an algebra to have a finite Gröbner basis.

## 2. THE PLAN OF CONSTRUCTION

Let  $\mathbb{A}$  be an algebra over a field  $\mathbb{K}$ . Fix a finite alphabet of generators  $\{a_1, \dots, a_N\}$ . A word in the alphabet of generators is called a *word in algebra*. An order on generators gives rise to the lex order on the set of all words.

The set of all words in the alphabet is a semigroup. The main idea of the construction is a realization of a universal Turing machine in the semigroup. We use the universal Turing machine constructed by Marvin Minsky in [Mi]. This machine has 7 states and 4-color tape. The machine can be completely defined by 28 instructions. Note that 27 of them have a form

$$(i, j) \rightarrow (L, q(i, j), p(i, j)) \text{ or } (i, j) \rightarrow (R, q(i, j), p(i, j)),$$

where  $0 \leq i \leq 6$  is the current machine state,  $0 \leq j \leq 3$  is the current cell color,  $L$  or  $R$  (left or right) is the direction of a head moving after execution of the current instruction,  $q(i, j)$  is the state after current instruction,  $p(i, j)$  is the new color of the current cell.

Thus, the instruction  $(2, 3) \rightarrow (L, 3, 1)$  means the following: “If the color of the current cell is 3 and the state is 2, then the cell changes the color to 1, the head moves one cell to the left, the machine changes the state to 3.

The last instruction is  $(4, 3) \rightarrow \text{STOP}$ . Hence, if the machine is in state 4 and the current cell has color 3, then the machine halts.

**Letters.** By  $Q_i$ ,  $0 \leq i \leq 6$  denote the current state of the machine. By  $P_j$ ,  $0 \leq j \leq 3$  denote the color of the current cell.

The action of the machine depends on the current state  $Q_i$  and current cell color  $P_j$ . Thus every pair  $Q_i$  and  $P_j$  corresponds to one instruction of the machine.

The instructions moving the head to the left (right) are called *left (right)* ones. Therefore there are *left pairs*  $(i, j)$  for the left instructions, *right pairs* for the right ones and instruction STOP for the pair  $(4, 3)$ .

All cells with nonzero color are said to be *non-empty cells*. We shall use letters  $a_1, a_2, a_3$  for nonzero colors and letter  $a_0$  for color zero. Also, we use  $R$  for edges of colored area. Hence, the word  $Ra_{u_1}a_{u_2} \dots a_{u_k}Q_iP_ja_{v_1}a_{v_2} \dots a_{v_k}R$  presents a full state of Turing machine.

We model head moving and cell painting using computations with powers of  $a_i$  (cells) and  $P_i$  and  $Q_i$  (current cell and state of the machine’s head).

Every defining relation (except one monomial relation  $Q_4P_3 = 0$ ) has the form  $tA = At$ , where the subword  $A$  contains no  $t$  letter. The multiplication of the word  $Ra_{u_1}a_{u_2} \dots a_{u_k}Q_iP_ja_{v_1}a_{v_2} \dots a_{v_k}R$  by  $t$  corresponds to one turn of the Turing machine.

## 3. UNIVERSAL TURING MACHINE

We use the universal Turing machine constructed by Minsky. This machine is defined by the following instructions:

$$\begin{aligned} (0, 0) &\rightarrow (L, 4, 1) \quad (0, 1) \rightarrow (L, 1, 3) \quad (0, 2) \rightarrow (R, 0, 0) \quad (0, 3) \rightarrow (R, 0, 1) \\ (1, 0) &\rightarrow (L, 1, 2) \quad (1, 1) \rightarrow (L, 1, 3) \quad (1, 2) \rightarrow (R, 0, 0) \quad (1, 3) \rightarrow (L, 1, 3) \\ (2, 0) &\rightarrow (R, 2, 2) \quad (2, 1) \rightarrow (R, 2, 1) \quad (2, 2) \rightarrow (R, 2, 0) \quad (2, 3) \rightarrow (L, 4, 1) \\ (3, 0) &\rightarrow (R, 3, 2) \quad (3, 1) \rightarrow (R, 3, 1) \quad (3, 2) \rightarrow (R, 3, 0) \quad (3, 3) \rightarrow (L, 4, 0) \end{aligned}$$

$(4, 0) \rightarrow (L, 5, 2)$   $(4, 1) \rightarrow (L, 4, 1)$   $(4, 2) \rightarrow (L, 4, 0)$   $(4, 3) \rightarrow \text{STOP}$   
 $(5, 0) \rightarrow (L, 5, 2)$   $(5, 1) \rightarrow (L, 5, 1)$   $(5, 2) \rightarrow (L, 6, 2)$   $(5, 3) \rightarrow (R, 2, 1)$   
 $(6, 0) \rightarrow (R, 0, 3)$   $(6, 1) \rightarrow (R, 6, 3)$   $(6, 2) \rightarrow (R, 6, 2)$   $(6, 3) \rightarrow (R, 3, 1)$

We use the following alphabet:

$$\{t, a_0, \dots, a_3, Q_0, \dots, Q_6, P_0, \dots, P_3, R\}$$

For every pair except  $(4, 3)$  the following functions are defined:  $q(i, j)$  is a new state,  $p(i, j)$  is a new color of the current cell (the head leaves it).

#### 4. DEFINING RELATIONS FOR THE NILPOTENCY QUESTION

Consider the following defining relations:

$$tR = Rt; \tag{4.1}$$

$$ta_k = a_k t; \quad 0 \leq k \leq 3 \tag{4.2}$$

$$ta_k Q_i P_j = Q_{q(i,j)} P_k a_{p(i,j)} t; \text{ for left pairs } (i, j) \text{ and } 0 \leq k \leq 3 \tag{4.3}$$

$$tQ_i P_j a_k = a_{p(i,j)} Q_{q(i,j)} P_k t; \text{ for right pairs } (i, j) \text{ and } 0 \leq k \leq 3 \tag{4.4}$$

$$tRQ_i P_j = RQ_{q(i,j)} P_0 a_{p(i,j)} t; \text{ for left pairs } (i, j) \tag{4.5}$$

$$tQ_i P_j R = a_{p(i,j)} Q_{q(i,j)} P_0 R t; \text{ for right pairs } (i, j) \tag{4.6}$$

$$Q_4 P_3 = 0. \tag{4.7}$$

The relations  $\langle 4.1 \rangle$ – $\langle 4.2 \rangle$  are used to move  $t$  from the left edge to the letters  $Q_i, P_j$  which represent the head of the machine. The relations  $\langle 4.3 \rangle$ – $\langle 4.6 \rangle$  represent the computation process.

Finally, the relation  $\langle 4.7 \rangle$  halts the machine.

#### 5. NILPOTENCY OF THE FIXED WORD AND MACHINE HALT

Let us call the word  $tRa_{u_1}a_{u_2}\dots a_{u_k}Q_iP_ja_{v_1}a_{v_2}\dots a_{v_k}R$  the *main word*. The main goal is to prove the following theorem:

**Theorem 5.1.** *The machine halts if and only if the main word is nilpotent in the algebra presented by the defining relations  $\langle 4.1 \rangle$ – $\langle 4.7 \rangle$ .*

First, we prove some propositions.

**Remark.** We use  $\text{sign} =$  for lexicographical equality and  $\text{sign} \equiv$  for equality in algebra.

Consider a full state of our Turing machine represented by the word

$$Ra_{u_1}a_{u_2}\dots a_{u_k}Q_iP_ja_{v_1}a_{v_2}\dots a_{v_k}R.$$

Suppose that  $U = a_{u_1}a_{u_2}\dots a_{u_k}$  and  $V = a_{v_1}a_{v_2}\dots a_{v_k}$ . Therefore  $U$  and  $V$  represent the colors of all cells on the Turing machine tape. We denote the full state of this machine as  $M(i, j, U, V)$ . Suppose that  $M(i', j', U', V')$  is the next state ( $M(i, j, U, V) \rightarrow M(i', j', U', V')$ ).

Consider a semigroup  $G$  presented by the defining relations  $\langle 4.1 \rangle$ – $\langle 4.7 \rangle$ . Suppose that  $W(i, j, U, V)$  is a word in  $G$  corresponding to machine state  $M(i, j, U, V)$ . (Actually  $W(i, j, U, V) = Ra_{u_1}a_{u_2}\dots a_{u_k}Q_iP_ja_{v_1}a_{v_2}\dots a_{v_k}R$ .)

**Proposition 5.1.** *If  $i = 4$  and  $j = 3$  then  $tW(i, j, U, V) \equiv 0$ . Otherwise, the following condition holds:  $tW(i, j, U, V) \equiv W(i', j', U', V')t$ .*

**Proof.** Consider the word  $tW(i, j, U, V) = tRUQ_iP_jVR$ . If  $i = 4$  and  $j = 3$  then we can apply relation  $\langle 4.7 \rangle$ . Otherwise, suppose that  $(i, j)$  is a left pair.

If  $U$  is an empty word then  $tW(i, j, U, V) = tRUQ_iP_jVR$ . Hence we can apply relation  $\langle 4.5 \rangle$  to obtain  $tRUQ_iP_jVR \equiv RQ_{q(i,j)}P_0a_jtVR$ . Using  $\langle 4.1 \rangle$  and  $\langle 4.2 \rangle$  we finally have

$$tRUQ_iP_jVR \equiv RQ_{q(i,j)}P_0a_jtVR \equiv RQ_{q(i,j)}P_0a_jVRt.$$

According to the definition of  $q(i, j)$  and  $p(i, j)$ , this word corresponds to the next state of the machine.

If  $U$  is not an empty word, we can write  $U = U_1a_k$  for some  $k$ . We use the relations  $\langle 4.1 \rangle$  and  $\langle 4.2 \rangle$  and obtain that  $tRUQ_iP_jVR \equiv RU_1ta_kQ_iP_jVR$ . Further, we use relation  $\langle 4.3 \rangle$ :  $RU_1ta_kQ_iP_jVR \equiv RU_1Q_{q(i,j)}P_ka_{p(i,j)}VRt$ . The last word corresponds to the next state of the machine.

The case of right pair is similar, we just use relations  $\langle 4.6 \rangle$  and  $\langle 4.4 \rangle$ .  $\square$

**Proposition 5.2.** *Let us move all the words from the relations  $\langle 4.1 \rangle$ – $\langle 4.7 \rangle$  to the left-hand side. Consider the LEX order:  $\{t, a_0, \dots, a_3, Q_0, \dots, Q_6, P_0, \dots, P_3, R\}$ . The left-hand sides of the obtained equalities comprise a Gröbner basis in the ideal generated by them.*

**Proof.** Note that every left-hand side contains a leading monomial. There is no such word that begins some leading monomial in the basis and ends some other leading monomial.  $\square$

**Proposition 5.3.** *The following statements are equivalent:*

- (i) *The Turing machine described above begins with the state  $M(i, j, U, V)$  and halts in several steps.*
- (ii) *There exists a positive integer  $N$  such that  $t^N RUQ_iP_jVR \equiv 0$ .*

**Proof.** First, prove that second statement is a consequence of the first one.

Suppose that  $M(i, j, U, V)$  transforms to  $M(4, 3, U', V')$  in one step. According to Proposition 5.1  $tW(i, j, U, V) \equiv W(4, 3, U', V')t$ . Then we can apply  $Q_4P_3 = 0$  by  $\langle 4.7 \rangle$  and obtain zero.

Suppose that the statement is true for  $m$  (and fewer) steps. Let the machine begin with state  $M(i, j, k, n)$  and halt after  $m + 1$  step. Consider the first step in the chain. Let it be the step from  $M(i, j, U, V)$  to  $M(i', j', U', V')$ . Apply Proposition 5.1 for this step. Hence  $tRUQ_iP_jVR \equiv RU'Q_{i'}P_{j'}VRt$ .

The machine started in the state  $M(i', j', U', V')$  halts in  $m$  steps. Using induction we complete the proof.

Now let us prove that the first statement is a consequence of the second one.

Consider the machine which begins with the state  $M(i, j, U, V)$  and does not halt. Let  $N$  be the minimal positive integer such that  $t^N RUQ_iP_jVR \equiv 0$ . Let us apply some relations to the word  $t^N RUQ_iP_jVR$  and obtain word  $W$ .

**Definition 5.1.** Let us delete all letters  $t$  from  $W$ . We obtain word  $S(W)$ . We say that  $S(W)$  is *structure of the word  $W$* .

It is clear that for any structure  $S(W)$  there exists a corresponding state of the machine. Relations  $\langle 4.1 \rangle$  and  $\langle 4.2 \rangle$  do not change the structure, relations  $\langle 4.3 \rangle$  –  $\langle 4.6 \rangle$  turn structure to the next (or previous) state of the machine. If  $t^N RUQ_iP_jVR \equiv 0$  then it is possible to transform  $t^N RUQ_iP_jVR$  to the form which contains  $Q_4P_3$ . Thus, tracing the whole chain of transformations we can write the sequence of  $n$  consecutive states of machine with the last state corresponding to  $Q_4P_3$ . Therefore the machine halts after  $n$  steps.  $\square$

Now we are ready to prove the theorem above.

**Theorem 5.2.** *Consider an algebra  $A$  presented by the defining relations  $\langle 4.1 \rangle$ – $\langle 4.7 \rangle$ . The word  $tRUQ_iP_jVR$  is nilpotent in  $A$  if and only if machine  $M(i, j, U, V)$  halts.*

**Proof.** Suppose that  $(tRUQ_iP_jVR)^n \equiv 0$ . The structure of this word corresponds to a row of  $n$  separate machines. Using relations we can transform some machine to the next or to the previous state. Thus if we obtain  $Q_4P_3$  for some machine, we can conclude that this machine halts after several steps. Therefore  $M(i, j, U, V)$  halts.

Suppose that  $M(i, j, U, V)$  halts. Then  $t^nRUQ_iP_jVR \equiv 0$  for some minimal  $n$ . We can obtain  $(tRUQ_iP_jVR)^n \equiv At^nRUQ_iP_jVR$  (for some word  $A$ ) by using Proposition 5.1 several times. Therefore  $(tRUQ_iP_jVR)^n \equiv 0$ .  $\square$

Since the halting problem cannot be algorithmically solved, the nilpotency problem in algebra  $A$  is algorithmically unsolvable.

## 6. DEFINING RELATIONS FOR A ZERO DIVISORS QUESTION

We use the following alphabet:

$$\{t, s, a_0, \dots, a_3, Q_0, \dots, Q_6, P_0, \dots, P_3, L, R\}.$$

For every pair except  $(4, 3)$  the following functions are defined:  $q(i, j)$  is a new state,  $p(i, j)$  is a new color of the current cell (the head leaves it).

Consider the following defining relations:

$$tL = Lt; \tag{6.1}$$

$$ta_k = a_k t; \quad 0 \leq k \leq 3 \tag{6.2}$$

$$sR = Rs; \tag{6.3}$$

$$sa_k = a_k s; \tag{6.4}$$

$$ta_k Q_i P_j = Q_{q(i,j)} P_k a_{p(i,j)} s; \text{ for left pairs } (i, j) \text{ and } 0 \leq k \leq 3 \tag{6.5}$$

$$tQ_i P_j a_k = a_{p(i,j)} Q_{q(i,j)} P_k s; \text{ for right pairs } (i, j) \text{ and } 0 \leq k \leq 3 \tag{6.6}$$

$$tLQ_i P_j = LQ_{q(i,j)} P_0 a_{p(i,j)} s; \text{ for left pairs } (i, j) \tag{6.7}$$

$$tQ_i P_j R = a_{p(i,j)} Q_{q(i,j)} P_0 R s; \text{ for right pairs } (i, j) \tag{6.8}$$

$$Q_4 P_3 = 0; \tag{6.9}$$

The relations  $\langle 6.1 \rangle$ – $\langle 6.2 \rangle$  are used to move  $t$  from the left edge to the letters  $Q_i, P_j$  which present the head of the machine. Similarly, the relations  $\langle 6.3 \rangle$ – $\langle 6.4 \rangle$  are used to move  $s$  from the letter  $Q_i, P_j$  to the right edge. The relations  $\langle 6.5 \rangle$ – $\langle 6.8 \rangle$  represent the computation process. Here we use relations of the form  $tU = Vs$ .

Finally, the relation  $\langle 6.9 \rangle$  halts the machine.

## 7. ZERO DIVISORS AND MACHINE HALT

Let us call the word  $La_{u_1}a_{u_2}\dots a_{u_k}Q_iP_ja_{v_1}a_{v_2}\dots a_{v_k}R$  the *main word*. The main goal is to prove the following theorem:

**Theorem 7.1.** *The machine halts if and only if the main word is a zero divisor in the algebra presented by the defining relations  $\langle 6.1 \rangle$ – $\langle 6.9 \rangle$ .*

Consider a full state of our Turing machine represented by the word

$$La_{u_1}a_{u_2}\dots a_{u_k}Q_iP_ja_{v_1}a_{v_2}\dots a_{v_k}R.$$

Suppose that  $U = a_{u_1}a_{u_2}\dots a_{u_k}$  and  $V = a_{v_1}a_{v_2}\dots a_{v_k}$ . Therefore  $U$  and  $V$  represent the colors of all cells on the Turing machine tape. We denote the full state of this

machine as  $T(i, j, U, V)$ . Suppose that  $T(i', j', U', V')$  is the next state ( $T(i, j, U, V) \rightarrow T(i', j', U', V')$ ).

Consider a semigroup  $S$  presented by the defining relations (6.1)–(6.9). Suppose that  $F(i, j, U, V)$  is a word in  $S$  corresponding to machine state  $T(i, j, U, V)$ .

**Proposition 7.1.** *If  $i = 4$  and  $j = 3$  then  $tF(i, j, U, V) \equiv 0$ . Otherwise, the following condition holds:  $tF(i, j, U, V) \equiv F(i', j', U', V')s$ .*

**Proof.** Consider the word  $tF(i, j, U, V) = tLUQ_iP_jVR$ . If  $i = 4$  and  $j = 3$  then we can apply relation (6.9). Otherwise, suppose that  $(i, j)$  is a left pair.

If  $U$  is an empty word then  $tF(i, j, U, V) = tLQ_iP_jVR$ . Hence we can apply relation (6.7) to obtain  $tLQ_iP_jVR \equiv LQ_{q(i,j)}P_0a_jsVR$ . Using (6.3) and (6.4) we finally have

$$tLQ_iP_jVR \equiv LQ_{q(i,j)}P_0a_jsVR \equiv LQ_{q(i,j)}P_0a_jVRs.$$

According to the definition of  $q(i, j)$  and  $p(i, j)$ , the word  $LQ_{q(i,j)}P_0a_jVR$  corresponds to the next state of the machine.

If  $U$  is not an empty word, we can write  $U = U_1a_k$  for some  $k$ . We use the relations (6.1) and (6.2) and obtain that  $tLUQ_iP_jVR \equiv LU_1ta_kQ_iP_jVR$ . Further, we use relation (6.5):  $LU_1ta_kQ_iP_jVR \equiv LU_1Q_{q(i,j)}P_ka_{p(i,j)}VRs$ . The word  $LU_1Q_{q(i,j)}P_ka_{p(i,j)}VR$  corresponds to the next state of the machine.

The case of a right pair is similar, we just use relations (6.8) and (6.6).  $\square$

**Proposition 7.2.** *Let us move all the words from relations (6.1)–(6.9) to the left-hand side. Consider the LEX order:  $\{t, s, a_0, \dots, a_3, Q_0, \dots, Q_6, P_0, \dots, P_3, L, R, \}$ . The left-hand sides of the obtained equalities comprise a Gröbner basis in the ideal generated by them.*

**Proof.** Note that every left-hand side contains a leading monomial. There is no such word that begins some leading monomial in the basis and ends some other leading monomial.  $\square$

**Proposition 7.3.** *The following statements are equivalent:*

- (i) *The Turing machine described above begins with the state  $T(i, j, U, V)$  and halts in several steps.*
- (ii) *There exists a positive integer  $N$  such that  $t^N LUQ_iP_jVR \equiv 0$ .*

**Proof.** First, prove that second statement is a consequence of the first one.

Suppose that  $T(i, j, U, V)$  transforms to  $T(4, 3, U', V')$  in one step. According to Proposition 7.1  $tF(i, j, U, V) \equiv F(4, 3, U', V')t$ . Then we can apply  $Q_4P_3 = 0$  by (6.9) and obtain zero.

Suppose that the statement is true for  $m$  (and fewer) steps. Let the machine begin with state  $T(i, j, k, n)$  and halt after  $m + 1$  step. Consider the first step in the chain. Let it be the step from  $T(i, j, U, V)$  to  $T(i', j', U', V')$ . Apply Proposition 7.1 for this step. Hence  $tLUQ_iP_jVR \equiv LU'Q_{i'}P_{j'}V'R_s$ .

The machine started in the state  $T(i', j', U', V')$  halts in  $m$  steps. Using induction we complete the proof.

Now let us prove that the first statement is a consequence of the second one.

Consider the machine which begins with the state  $T(i, j, U, V)$  and does not halt. Let  $N$  be the minimal positive integer such that  $t^N LUQ_iP_jVR \equiv 0$ . Let us apply some relations to the word  $t^N LUQ_iP_jVR$  and obtain word  $F$ .

**Definition 7.1.** Let us delete all letters  $t$  and  $s$  from  $F$ . We obtain word  $S(F)$ . We say that  $S(F)$  is *structure of the word  $F$* .

It is clear that for any structure  $S(F)$  there exists a corresponding state of the machine. Relations  $\langle 6.1 \rangle - \langle 6.4 \rangle$  do not change the structure, relations  $\langle 6.5 \rangle - \langle 6.8 \rangle$  turn structure to the next (or previous) state of the machine. If  $t^N LUQ_i P_j VR \equiv 0$  then it is possible to transform  $t^N RUQ_i P_j VR$  to the form which contains  $Q_4 P_3$ . Thus, tracing the whole chain of transformations we can write the sequence of  $n$  consecutive states of machine with the last state corresponding to  $Q_4 P_3$ . Therefore the machine halts after  $n$  steps.  $\square$

**Proposition 7.4.** *If  $Xt \equiv 0$  in  $S$ , then  $X \equiv 0$ . If  $sX \equiv 0$  in  $S$ , then  $X \equiv 0$ .*

**Proof.** Suppose that we apply some relations and transform  $Xt$  to zero.

We say that the letter  $t$  is *almost last* if the word has the form  $Y_1 t Y_2$ , and  $Y_2$  contains  $a_k$  and  $L$  letters only. Note that if an almost last  $t$ -letter occurs in some relation then this relation is  $\langle 6.1 \rangle$  or  $\langle 6.2 \rangle$ . Therefore that  $t$ -letter is always almost last. It is clear that an almost last  $t$ -letter always exists in every word which is equivalent to  $Xt$ . Since an almost last  $t$ -letter never participates in relations  $\langle 6.3 \rangle - \langle 6.9 \rangle$ , we can situate it on the right edge of the word  $Xt$  while we use our relations. We did not use the  $t$ -letter, and therefore we can do the same with the word  $X$ .

Similarly we can prove that if  $sX \equiv 0$  then  $X \equiv 0$ .  $\square$

**Proposition 7.5.** *If  $Xt^n \equiv 0$  in  $S$ , then  $X \equiv 0$ . If  $s^n X \equiv 0$  in  $S$ , then  $X \equiv 0$ .*

**Proof.** We can prove this by induction.  $\square$

Now we are ready to prove the theorem above.

**Theorem 7.2.** *Consider an algebra  $H$  presented by the defining relations  $\langle 6.1 \rangle - \langle 6.9 \rangle$ .*

*The word  $LUQ_i P_j VR$  is a zero divisor in the algebra  $H$  if and only if machine  $T(i, j, U, V)$  halts.*

**Proof.** Suppose that machine  $T(i, j, U, V)$  halts. Using Proposition 7.3 we have  $t^N LUQ_i P_j VR \equiv 0$  for some positive integer  $N$ . Thus, the word  $LUQ_i P_j VR$  is a zero divisor.

Let  $XLUQ_i P_j VRY \equiv 0$  for some algebra elements  $X, Y \neq 0$ . Suppose that  $X, Y$  are some words.

Note that  $L$  and  $R$  letters cannot disappear from the word. Hence we can divide our word into three parts: to the left of  $L$ , to the right of  $R$ , and between  $L$  and  $R$ . There is only one relation which can turn the word  $XLUQ_i P_j VRY$  to zero:  $Q_4 P_3 = 0$ . Thus this subword  $Q_4 P_3$  can appear in three possible parts of the word. Note that only  $t$  letters can pass through  $L$  and only  $s$  letters can pass through  $R$ . Every relation can change nothing in the area to the left side of  $L$  and to the right side of  $R$ , except  $t$  and  $s$ -letters occurrences. Therefore if  $Q_4 P_3$  appears to the left of  $L$ , then  $Xs^n \equiv 0$ . Using Proposition 7.5 we obtain a contradiction:  $X \equiv 0$ . Similarly if  $Q_4 P_3$  appears to the right of  $R$ , then  $Y \equiv 0$ . Thus  $Q_4 P_3$  appears between  $L$  and  $R$ .

Consider the *structure* of the word  $LUQ_i P_j VR$ . For any structure of a word equivalent to  $LUQ_i P_j VR$  there exists a corresponding state of the machine. Since only  $t$  letters can pass through  $L$  and only  $s$  letters can pass through  $R$ , we can change the structure of the word  $LUQ_i P_j VR$  by turn to the next or the previous machine state. If  $Q_4 P_3$  appears between  $L$  and  $R$  then we can obtain a STOP state. Thus the machine  $T(i, j, U, V)$  halts.

Now let us consider the general case:  $X, Y$  are some algebra elements. Suppose that  $X = X_1 + \dots + X_n, Y = Y_1 + \dots + Y_m, X_i, Y_i$  are monomials and we use the minimal possible  $n$ . We have  $XLUQ_i P_j VRY = \sum_{k,l} X_k LUQ_i P_j VRY_l$ . We can consider our defining relations as reductions and use them to find the Gröbner basis of every term  $X_k LUQ_i P_j VRY_l$ . If one of these terms can be reduced to zero we can apply the case considered above. Otherwise, we have  $X_k LUQ_i P_j VRY_l \equiv X'_k LUQ_i P_j VRY'_l$ , and  $X'_k LUQ_i P_j VRY'_l$  cannot be reduced.

Let us fix the  $t$ 's at the end of the  $X'_k$  words:  $X'_k = X''_k t^{q_i}$ . These are lexicographical equalities and  $q_i \geq 0$ .

Since  $\sum_{k,l} X''_k t^{q_i} LUQ_i P_j VRY'_l \equiv 0$ , this sum (in the reduced form) can be separated into several sets of similar monomials. Consider one of these sets:  $\sum_{1 \leq u \leq h} X''_u t^{x_u} LUQ_i P_j VRY'_u$ . If these monomials are similar then all  $X''_u$  must be also similar. Further, all  $x_u$  must be the same and all  $Y'_u$  must be similar in the reduced form. Therefore, we can choose smaller  $n$  in the presentation  $X = X_1 + \dots X_n$ .

This contradiction completes the proof.  $\square$

Since the halting problem cannot be algorithmically solved, the zero divisors problem in algebra  $H$  is algorithmically unsolvable.

#### REFERENCES

- [Be] Bergman, G. *The diamond lemma for ring theory*. Adv. Math., (1978), 29, 2, 178–218.
- [Bo] Bokut, L. *Unsolvability of the equality problem and subalgebras of finitely presented lie algebras*. Izvestiya Akad. Nauk SSSR. **36**:6 (1972), 1173–1219
- [BC] Bokut, L., Chen Y. *Gröbner-Shirshov bases and PBW theorems* J. Sib.Fed. Univ. Math. Phys., 2013, Volume 6, Issue 4, 417–427
- [BK] Bokut, L., Kukin G. *Undecidable algorithmic problems for semigroups, groups and rings. (Russian)* Translated in J. Soviet Math. **45** (1989), no. 1, 871–911. Itogi Nauki i Tekhniki, Algebra. Topology. Geometry, Vol. 25 (Russian), 3–66, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, (1987).
- [I1] Iyudu, N. *Algorithmical solvability of zero divisors problem in one class of algebras*. Pure and Applied Math., (1995), 2, 1, 541–544.
- [I2] Iyudu, N. *Standart bases and property solvability in the algebras defined by relations*. Dissertation — Moscow, (1996), 73.
- [IP] Ivanov-Pogodaev, I. *An algebra with a finite Gröbner basis and an unsolvable problem of zero divisors*. J. Math. Sci. (N. Y.) 152 (2008), no. 2, 191–202
- [KB] Kanel-Belov, A. *Linear recurrence relations on tree*. Math. zametki, 78, N5, 643–651.
- [KBBL] Kanel-Belov, A.; Borisenko V.; Latyshev V. *Monomial Algebras*. NY. Plenum (1997)
- [L] Latyshev, V. *On the recognizable properties of associative algebras*. Special vol. J.S.C.: On computational aspects of commutative algebras. London: Acad. Press, (1988), 237–254.
- [Ma] Markov, A. *The impossibility of certain algorithms in the theory of associative systems. (Russian)* Doklady Akad. Nauk SSSR, **55** N7, 1947, 587–590
- [Mi] Minsky, M. *Computation: Finite and Infinite Machines* (1967)
- [N1] Novikov, P., *On algorithmic unsolvability of the problem of identity. (Russian)* Doklady Akad. Nauk SSSR **85** N4, (1952). 709–712.
- [N2] Novikov, P., *On the algorithmic unsolvability of the word problem in group theory*. Trudy Mat. Inst. im. Steklov. no. 44. Izdat. Akad. Nauk SSSR, Moscow, (1955), 3–143
- [Po] Post E., *Recursive unsolvability of a problem of Thue*. J. Symb. Logic, 12, N1, 1947, 1–11
- [Pi1] Piontkovsky, D. *Gröbner base and coherence of monomial associative algebra*. Pure and Applied Math., (1996), 2, 2, 501–509.
- [Pi2] Piontkovsky, D. *Noncommutative Gröbner bases, coherence of monomial algebras and divisibility in semigroups* Pure and Applied Math., (2001), 7, 2, 495–513.
- [Pi3] Piontkovsky, D. *On the Kurosh problem in varieties of algebras*. J. Math. Sci., New York 163, No. 6, 743–750 (2009); translation from Fundam. Prikl. Mat. 14, No. 5, 171–184 (2008).
- [Pi4] Piontkovsky, D. *Graded algebras and their differential graded extensions*. J. Math. Sci., New York 142, No. 4, 2267–2301 (2007); translation from Sovrem. Mat. Prilozh. 30, 65–100 (2005).
- [Sh] Shirshov, A. *Some algorithmic problems for Lie algebras* Sib. mat. journal, (1962), vol 3 N2, 292–296
- [U] Ufnarovsky, V. *Combinatorial and asymptotic methods in algebra*. Itogi nauki i tekhniki, Modern problems of pure math. : VINITI, (1990), 57, 5–177.

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